Exercise 43

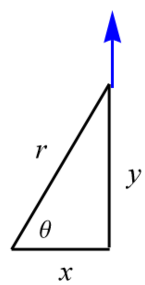
A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

Solution

Draw a schematic of the rocket launch at a certain time.



$$\left.\frac{dy}{dt}\right|_{y=3000} = 600\ \frac{\text{ft}}{\text{s}}$$

$$\frac{dx}{dt} = 0\ \frac{\text{ft}}{\text{s}}$$

Part (a)

The aim here is to find dr/dt. Start with the Pythagorean theorem, which relates the sides of a right triangle.

$$r^2 = x^2 + y^2$$

Take the derivative of both sides with respect to time by using the chain rule.

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + y^2)$$

$$2r \cdot \frac{dr}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$= x(0) + y\frac{dy}{dt}$$

$$= y\frac{dy}{dt}$$

Solve for dr/dt.

$$\begin{aligned}\frac{dr}{dt} &= \frac{y}{r}\frac{dy}{dt} \\ &= \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}\end{aligned}$$

Therefore, the rate of change of the distance from the television camera to the rocket with respect to time at the moment when the rocket's speed is 600 ft/s and height is 3000 feet is

$$\left.\frac{dr}{dt}\right|_{\substack{x=4000\\y=3000}} = \frac{(3000)}{\sqrt{(4000)^2+(3000)^2}}(600) = 360\ \frac{\text{ft}}{\text{s}}.$$

Part (b)

Use a trigonometric function to relate the angle θ with convenient sides of the triangle.

$$\cos\theta = \frac{x}{r}$$

Take the derivative of both sides with respect to time by using the chain rule.

$$\begin{aligned}\frac{d}{dt}(\cos\theta) &= \frac{d}{dt}\left(\frac{x}{r}\right) \\ (-\sin\theta)\cdot\frac{d\theta}{dt} &= \frac{\left(\frac{dx}{dt}\right)r - \left(\frac{dr}{dt}\right)x}{r^2} \\ \left(-\frac{y}{r}\right)\frac{d\theta}{dt} &= \frac{1}{r}\frac{dx}{dt} - \frac{x}{r^2}\frac{dr}{dt} \\ &= \frac{1}{r}(0) - \frac{x}{r^2}\frac{dr}{dt} \\ &= -\frac{x}{r^2}\frac{dr}{dt}\end{aligned}$$

Solve for $d\theta/dt$.

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{x}{yr}\frac{dr}{dt} \\ &= \frac{x}{y\sqrt{x^2+y^2}}\frac{dr}{dt}\end{aligned}$$

Therefore, the rate of change of the camera's angle of elevation with respect to time when the rocket's speed is 600 ft/s and height is 3000 feet is

$$\left.\frac{d\theta}{dt}\right|_{\substack{x=4000\\y=3000}} = \frac{4000}{(3000)\sqrt{(4000)^2+(3000)^2}}(360) = \frac{12}{125}\ \frac{\text{rad}}{\text{s}} = 0.096\ \frac{\text{rad}}{\text{s}}.$$